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Computable Chain-reachable Mapping of Linear and Quadratic Backward SOR Iteration for Newton Operator.

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ABSTRACT

The contraction mapping of Linear and Quadratic backward SOR methods for Newton operator in the nonlinear system of equation in real floating point arithmetic is presented. It is showed that if the computable reachable set of linear backward SOR method is lower chain-reachable to the outer computable chain-reachable Quadratic SOR method, the quadratic backward SOR method is not only finer in topology but also faster than backward linear SOR method if the arithmetic computational complexity involved in the execution of backward quadratic SOR is overlooked. This was demonstrated by a numerical example with the two methods where quadratic backward SOR method with Newton operator is showed to have superiority over the linear backward SOR with Newton operator. The computed results for the two methods were compared with results earlier obtained from Uwamusi where interval Gauss-Siedel method was used.

Keyword : nonlinear system, newton method, Brouwer fixed point theorem, Hahn –Banach extension theorem, SOR iteration matrix

MSC 2010 Category: 15A09, 15A 29, 97 I 20, 34G20

INTRODUCTION

The paper describes convergence speed between the linear and quadratic backward SOR iteration methods driven by Newton method in real floating point arithmetic for approximating solution to nonlinear system of equation [1] and [2] provided the analytic derivative of the function

$$F(x)=0, \tag{1.1}$$

is easily available.

This means that there exists $F: D \subset R^n \to R^n$ and, $x \in D$, for which the Frechet derivative in an open ball $S = S\left(x^*, r\right) \subset D$ remains valid. Newton method is attractive for the solution of nonlinear system (1.1) because of its global convergence for any choice of $x^{(0)} \in D$. Abstractly, Newton method is given by the equation

$$x^{(k+1)} = x^{(k)} - \frac{f(x^{(k)})}{f^{/}(x^{(k)})}, \ (k = 0,1,...,) \tag{1.2}$$

whose sequence of iterates, converges to the desired solution x^*.

Fundamentally, method (1.1) is hardly solved in the form it is presented rather, we often transform to an equivalent linear system

$$As = -b \quad , \tag{1.3}$$

where the matrix A is assumed to be a non singular Jacobian matrix. The solution in general for system 1.1 is given by

$$x_i^{(k+1)} = x_i^{(k)} + s_i^{(m)} \text{ (k=0,1,…,m=0,1,…, i=1,2,…,n)} \tag{1.4}$$

where $s_i^{(m)}$ is in form 1.3.This means, there is a closed balanced absorbent subset $S \subset D$ of a linear topological space that is ultra-barrelled for which the sequence $\{s_i^{(m)}\}_{m=1}^{\infty}$, a closed balanced absorbent subsets of E for the graph of F that is necessarily closed in the product topology. The sequence $\{s_i^{(m)}\}_{m=1}^{\infty}$ is a defining sequence for S wherefrom, any inductive limit of countable globally convex ultra-barrel space generated by Newton method is convex, [3].

By further adoption of Miranda's theorem on the function F would yield that $x+s$ is a base of neighbourhoods of x_0 in (E,D) for which a fixed point theory for contraction mapping $f(x) = x$ holds. The proof of this is well known in many literature Texts, as result we omit.

The quadratic convergence of Newton method is that for any $0 < \alpha^* < 1$ there holds the estimate

$$\left\| x^* - x^{(k+1)} \right\|_\infty^2 \le \alpha^* \left\| x^* - x^{(k)} \right\|_\infty^2, \ (k = 0,1,...,) \tag{1.5}$$

A philosophical consideration will be '' if Newton operator of equation 1.4 is quadratically convergent, what is the nature of shrinkable neighbourhoods in Hausdorff space?'' It is known that Newton operator is monotone and has a shrinkable base of balanced neighbourhoods in (E,D) for which $\alpha_i s_i^{(m)} \in D_i$, *where* $0 \le \alpha_i < 1$. As the base is shrinking, it forces the sequence $\{\alpha_i s_i^{(m)}\}_{m=1}^{\infty}$ to converge to zero vector, as k approaches infinity, a consequence of Banach contraction mapping of a fixed point.

In [4], it was showed that Hansen-Sengupta method diverges if there were multiple paths simultaneously crossing a single point. This was demonstrated on a Trapezoidal Newton method. In other word, the shrinkable base neighbourhood failed to hold thereof , an indication of not only leading to stagnation point other than the solution x^* being sought but divergence to infinity. It was a motivation of the above preambles that adoption of the following theorems will be found useful as a tool in our work.

Theorem 1.1, (Brouwer, [5]). Let D be a convex and compact subset of R^n and, int $(D) \ne 0$ .Then every continuous mapping $G : D \to D$ has at least one fixed point $x^* \in D$, i.e., a point with $x^* = G(x^*)$.

A slight generalization of the theorem 1.1 can be found in Neumaier [5].

Theorem 1.2 , [5]. Let D be a convex and compact subset of R^n with int $(D) \ne 0$ and let $G : D \ne 0$ and suppose that $G : D \to P(D)$ be c- continuous. Then there is some point $x^* \in D$ with

$x^* \in G(x^*)$.

W e noted that when A is a Lipschitz set for F on $, D$ then F is Lipschitz continuous on D with Lipschitz constant $\alpha = \sup\{\|A\|_2\}$ as earlier stated in equation 1.5.

The remaining section in the paper is arranged as follows. In section 2, a class of SOR iteration method feasible in Newton operator is discussed. The aforementioned linear and quadratic backward SOR methods make use of relaxation parameter in their calculations, a brief review for the construction of over relaxation parameter ω in the interval $[1,2]$ was again visited in section 3.Section 4 gives numerical illustration of the presented methods and then conclusion is drawn at end of the paper based on our findings.

MATERIALS AND METHODS

As stated earlier in the beginning of this paper, Newton method for nonlinear system consists of successive linearization given in equation 1.3 where $A \in L(R^n), b \in R^n$, and $s \in R^n$ is to be found. In a well organized sense, the generalized class of stationary linear iterative solver to which equation 1.3 conforms is in the form:

$$s_i^{(m+1)} = Gs_i^{(m)} + c, \ (m = 0,1,..., i = 1,2,...,.) \tag{2.1}$$

where $s_1^{(0)}$ is arbitrary, and for some non singular matrix H, there exists a splitting matrix $A = H - (H - A)$ such that:

$$G = I - H^{-1}A, \ \ c = H^{-1}b \tag{2.2}$$

The matrix H appearing in equation 2.2 is a preconditioning matrix, further reference on the matrix H can be found in [6] and [7].

As a result we then form the convergent iteration sequence for the linear system defined by the equation

$$Hs_i^{(m+1)} = (H - A)s_i^{(m)} + b \text{ , (m=0,1, 2,..,i=1,2,…,.)} \tag{2.3}$$

Where a convergent sequence $\{s_i^{(m)}\}_{m=0}^{\infty}$ of vector iterates can be constructed provided regularity conditions for the matrix A are fulfilled.

Equation 2.3 usually lead to various matrix splitting [8] namely, taking:

$H = I \Rightarrow$ the Richardson method.

$H = D \Rightarrow$ the block Jacobi preconditioner

$H = \dfrac{1}{2-\omega}\left(\dfrac{1}{\omega}D - L\right)\left(\dfrac{1}{\omega}D^{-1}\right)\left(\dfrac{1}{\omega}D - U\right) \Rightarrow$ the symmetric successive over relaxation preconditioner

$H = (D - \omega L) \Rightarrow$ the SOR preconditioner

Because equation 2.1 is a stationary matrix iterative method convergence will be enhanced the faster the product $H^{-1}A$ approximates identity matrix. Following this discussion there holds:

If the linear backward SOR method as a reachable set is lower computable at what iterative point is it equal to outer computable Quadratic Backward SOR method in the chain reachable set? This inspires the following theorem.

Theorem 2.1,[9]. ''It is possible to compute lower approximations to the reachable set of a lower-semi-continuous system, and outer approximation to the chain-reachable set of an upper semi-continuous system if this set is compact. It is impossible to compute arbitrary –precision approximations to the reachable set of a continuous system if the closure of reachable set does not equal the chain reachable set.''

We situate theorem 2.1 with well known [3] Hahn-Banach extension theorem which relates that: if $E \subset R^n$ is a Hausdorff locally convex space and E_0 be a linear subspace of E, then any continuous linear functional on E_0 has a continuous linear extension to all of E provided that non zero functional of equation 1.1 is not exotic.

The concept of ε-chain is now defined which relates that if (X,d) is a metric space and $f : X \to X$ is a multivalued map, a sequence of points $x_0, x_1, ..., x_n$ is an ε-chain if there exist $s_1, s_2, ..., s_n \in X$ with $s_{n+1} \in f(x_i)$ such that $d(s_{i+1}, x_{i+1}) < \varepsilon$ for i=0,1,…,n-1. Thus a point x is chain reachable from X_0 if there is an ε-chain from X_0 to $X \; \forall \varepsilon > 0$. This is the fulcrum in which the equation 2.4 is built.

To steer our discussion in the right senses, the quadratic functional iteration [10] for which quadratic SOR method is applicable is now constructed in the form:

$$s_i^{(m+1)} = G^2 s_i^{(m)} + Gc + c, \; (m = 0,1,...,, i = 1,2,....,.) \tag{2.4}$$

Equation 2.4 is a stationary one point method with double over head cost. Practically, we now present the application of stated method as promised. First consider the well known linear backward SOR method which is in the form:

$$s_i^{(m+1)} = \frac{\omega_k}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij} s_j^{(m)} - \sum_{j=i+1}^{n} a_{ij} s_j^{(m+1)}\right) + (1-\omega_k) s_i^{(m)}, \; (m = 0,1,..., i = 1,2,..., n., k = 1,2,..,n)$$

(2.5)

In matrix notation, this equation 2.5 will take the form:

$$s^{(m+1)} = (D + \omega U)^{-1}((1-\omega)D - \omega L))s^{(m)} + (D + \omega U)^{-1}\omega b \tag{2.6}$$

Because of equation 2.6 we reformulate equation 2.4 for the quadratic backward SOR method and it will be given by the following equation

$$s^{(m+1)} = ((D + \omega D)^{-1}((1-\omega)D - \omega L))^2 s^{(m)} + (D + \omega U)^{-1}((1-\omega)D - \omega L)(D + \omega U)^{-1}\omega b, (m = 0,1,...,) \tag{2.7}$$

The point Jacobi iteration matrix in which method 2.5 subscribes has eigenvalues given by $\rho(D^{-1}(L+U)) < 1, \; \Rightarrow \|G\| < \rho(G) + \varepsilon$. Thus by a well known theorem, $\lim_{m\to\infty}(\|G^m\|) = \rho(G)$.

We move to compare the rate of convergence of the linear and quadratic backward SOR iteration matrices as they appeared in equations 2.6 and 2.7 respectively. Let $\Im_1$ and $\Im_2$ be two $n \times n$ real matrices which correspond respectively to linear and quadratic backward SOR iteration matrices, see e.g., [6] and[11]. We assume that there is in existence an integer m for which $\|\Im_1\| < 1$. We then define that

$$R\left(\Im_1^m\right) = -\ln\left[\left(\left\|\Im_1^m\right\|\right)^{\frac{1}{m}}\right] = -\ln\frac{\left\|\Im_1^m\right\|}{m} \tag{2.9}$$

is the average rate of convergence for m iterations of the matrix $\Im_1$. Now assuming by computation that $R\left(\Im_2^m\right) < R\left(\Im_1^m\right)$ then $\Im_2$ will be [6,7] considered iteratively faster for m iterations than $\Im_1$ with the same limit point, s^*.Further reference can be found in [1] and [11] based on definition of $Q-factors$. The $R(\Im^m)$ can be estimated by the Power method.

We measure average reduction factor per iteration for the m iterations in the successive error norms by the quantity which is defined as

$$\sigma = \left(\frac{\left\|e^{(m)}\right\|}{\left\|e^{(0)}\right\|}\right)^m, \text{ m=0,1,2,…,} \tag{2.10}$$

The term $e_m = s^{(m+1} - s^{(m)}$ is well defined. The expression $R\left(\Im^m\right)$ is bounded by a factor $\sigma \le \left\|\Im^m\right\|^{-m} = e^{-R\left(\Im^m\right)}$ provided $\left\|\Im^m\right\| < 1$.It is the exponential decay rate for a sharp upper bound for the average reduction factor σ per iteration. The number of iterations required to reduce the norm of the initial error e_0 by a factor η is defined to be $N_m = \left(R\left(\Im^m\right)\right)^{-1}$ and $\sigma^{N_m} \le \eta^{-1}$ always for any convergent SOR method [6].Because of high costs involved in terms of arithmetic calculations in obtaining $\Im^m$, the use of estimate for asymptotic convergence rate given by the relation $R(\Im) = \lim_{m\to\infty} R_m(\Im) = -\log\rho(\Im)$ for a non symmetric matrix may be advantageous.The same procedure applies to a problem leading to a symmetric matrix with

$$R_m(\Im) = -\frac{1}{m}\log\left\|\Im^m\right\|_2 = -\log\rho(\Im). \tag{2.11}$$

The stopping criterion for termination of iterative methods adopted in terms of relative residual is $r = b - Ax$ for sufficiently small enough r. As a test condition we noted that

$$\frac{\|r_k\|}{\|r_0\|} < \tau \quad , e_k = x^{(k)} - x^{(k-1)}; k = 1,2,...,.$$

where $\tau = tol$ the tolerance level the solution can allow and that $\dfrac{\left\|e^k\right\|}{\|e_0\|} \le K(A)\dfrac{\|r_k\|}{\|r_0\|}$.

Nevertheless, if we ignore the extra computational cost in the course of evaluating quadratic functional iteration per step for method 2.7 which was derived from equation 2.4 , and, instead, taking into consideration the gains in terms of finer topology it generates than that of linear backward SOR method, it can be analogous to [12] that the quadratic backward SOR method is faster than the linear SOR method as attested to in the presented figure 1 in section 4.We hope to present the analysis in a forth coming paper.

RESULT AND DISCUSSION

The construction of ω for SOR method And Convergence Analysis

The theoretical determination of ω can be found in [6], [7]). For easy accessibility we review here theoretical determination of ω as a crucial step in the implementation of the described methods. Let the iteration matrix for SOR in the determination of ω be denoted by $\Im_\omega$.Then we set as

$$\Im_\omega = \left(I - \omega D^{-1}L\right)^{-1}\left((1-\omega)I + \omega D^{-1}U\right) \tag{3.1}$$

The open set for spectrum of $\Im_\omega$is described by the relation $1\backslash\{\rho(\Im_\omega)\}$ such that $I - \Im_\omega \neq 0$ for which any $\omega \in (0,2)$ can be detailed. This is more so as the set $\{\lambda_i\}$ being the eigenvalues of SOR iteration matrix for which $\left|\prod_{i=1}^{n}\lambda_i\right| = \left|\det\left((1-\omega)I + \omega D^{-1}U\right)\right| = |1-\omega|^n$.

Since $|\lambda_i| \geq |1-\omega|$, it holds that $|1-\omega| < 1 \Rightarrow 0 < \omega < 2$. We derive the value of ω as follows.

The optimal relaxation parameter in the classical SOR theory is derived as follows:

The characteristic equation for the iteration matrix for the SOR is computed as

$$\text{Det }\left(\lambda I - \left(I - \omega D^{-1}L\right)\left((1-\omega)I + \omega D^{-1}U\right)\right) = 0 \tag{3.2}$$

Since $\det\left(I - \omega L\right) = 1$ for every value of ω, we may rewrite equation 3.2 in the form

$\det\left(\lambda I - \lambda\omega L - (1-\omega)I - \omega U\right) = \det\left((\lambda + \omega - 1)I - \lambda\omega L - \omega U\right).$

In matrix notation it is given by

$$\begin{vmatrix} (\lambda+\omega-1)a_{11} & -\omega a_{12} & \cdots & \omega a_{in} \\ -\lambda\omega a_{21} & (\lambda+\omega-1)a_{22} & \cdots & \omega a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ -\lambda\omega a_{n1} & -\lambda\omega a_{n2}\ \cdots & -\lambda\omega a_{n,n-1} & (\lambda+\omega-1)a_{n,n} \end{vmatrix} = 0 \tag{3.3}$$

In the sense of [6 and 7] ,the Jacobi iteration matrix $J = D^{-1}(L+U)$ is a crucial factor for determination of ω. For this, we define

$$\left(\rho(J)_{\omega_b}\right)^p = \left(p^p(p-1)^{1-p}(\omega_b - 1)\right) \tag{3.4}$$

The term $\rho(J)$ is the spectral radius of associated Jacobi matrix. Thus for p=2, the matrix J is consistently weakly cyclic of index 2 with real eigenvalues whose unique positive root of ω_b in equation 3.4 is given to be

$$\omega_b = \frac{2}{1 + \sqrt{1 - \rho^2(J)}} \tag{3.5}$$

The corresponding asymptotic convergence factor is

$$\rho\left(B_{\omega_{opt}}\right)=\frac{1-\sqrt{1-\rho^2}}{1+\sqrt{1-\rho^2}}. \tag{3.6}$$

Numerical Results

The sample numerical problem is taking from [13]:

$$f(x)=\begin{cases} 3x_1-\cos(x_2x_3)-0.5=0 \\ x_1^2-81(x_2+0.1)^2+\sin x_3+1.06=0 \\ e^{-x_1x_2}+20x_3+\dfrac{10\pi-3}{3}=0 \end{cases}$$

$$x^{(0)}=\left(x_1^{(0)},x_2^{(0)},x_3^{(0)}\right)=(0.1,0.1,-0.1)^T$$

Let m be the inner number of iterations required for the linear backward SOR method to attain its accuracy when tolerance for Newton iteration is met (see Table 1).

Tolerance for the outer iteration (Newton iteration) was fixed to be 1×10^{-4} while allowing variation for tolerance in the inner iteration (Backward SOR) methods), it was observed that at tolerance value of 10^{-15}, the results are the same for both linear Backward SOR and quadratic Backward SOR methods. This happened at the fifth successive iteration for linear Backward SOR method to attain the same accuracy of tolerance of 10^{-15} when it was at the third successive iteration for quadratic Backward SOR to attain the same tolerance of 10^{-15}.

As for outer iteration (the Newton iteration) ,the final results were obtained for both methods which use Newton method to approximate the zeros of $F(x)=0$ at the third iteration. The Tables 1 and 2 below explain further.

Table 1 showing numerical results.

TOL	Linear Backward SOR	M	Quadratic Backward SOR	M
10^{-5}	0.50000000000705 0000002441339 -0.52359846437847	1	0.50000000000708 0.00000000080787 -0.52359877498707	1
10^{-6}	0.50000000000708 0.00000000080736 -0.52359877498722	2	0.50000000000708 0.00000000080736 -0.52359877498722	1
10^{-8}	0.50000000000708 0.00000000077583 -0.52359877557722	3	0.50000000000708 0.00000000077579 -0.52359877557801	2

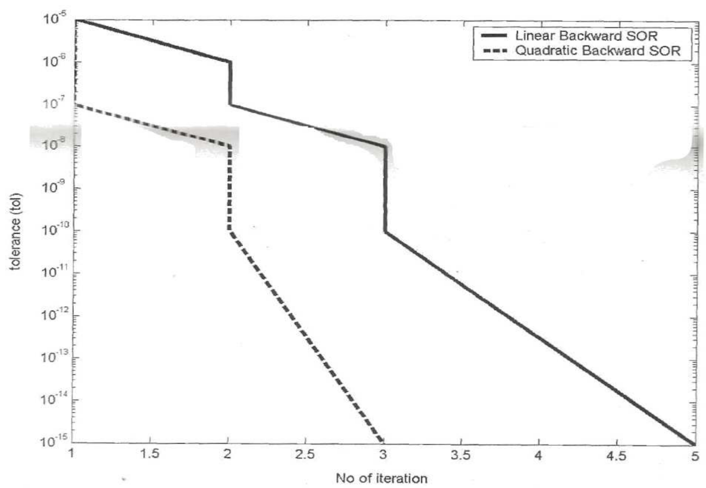
10^{-15}	0.50000000000708 0.00000000077579 -0.52359877557801	5	0.50000000000708 0.00000000077579 -0.52359877557801	3

Table 2 showing number of iterations versus Tol.

Linear M	Backward SOR K	Newton	Tol. SOR	Quadratic M	Backward SOR K
1	3	10^{-4}	10^{-5}	1	3
2	3	10^{-4}	10^{-6}	1	3
2	3	10^{-4}	10^{-7}	1	3
3	3	10^{-4}	10^{-8}	2	3
3	3	10^{-4}	10^{-9}	2	3
3	3	10^{-4}	10^{-10}	2	3
5	3	10^{-4}	10^{-15}	3	3

Fig 1 below shows graphical representation of tol against number of iterations.

Figure 1



CONCLUSION

The paper presented contraction mappings for both backward linear and quadratic SOR methods feasible in Newton operator in real floating point arithmetic to approximate the desired solution to nonlinear system of equation. It was discovered that quadratic Backward SOR method which uses Newton operator is not only finer in topology but also faster than the Classical Linear Backward SOR method which also uses Newton operator for the same purpose to approximate solution to nonlinear system of equation. This was done by plotting values of error per step (tolerance) against number of iterations as demonstrated in Figure 1 above. The computed results are displayed in Tables 1 and 2 for further illustration. Let us take note that the quadratic Backward SOR method converges if and only if its Linear Backward SOR method converges. The number of inner iteration at which Linear Backward SOR method equalled the inner iteration for Quadratic Backward SOR iteration formed the peak of our study, hence the name Chain-reachable mapping for the two different SOR methods. The significance of the study can be applied in Tidal Ocean waves or Tsunami waves, a significant branch of hydrodynamics in Water Mechanics Engineering where linearization through discretization of Partial Differential equation to linear system is imperative as a solution process other that method of Laplace transform. The proof of this will form a major part of forth coming publication as a follow up to this paper. The presented results for the two methods in floating point arithmetic are in close agreement with results earlier obtained in [13] where interval Guass-Siedel arithmetic operations were applied. In [13] for instance, we noted that accelerating interval operations by Sucessive Over-relaxation method was not worth the trouble so its implementation in the interval counterpart was carefully ignored at that time. It is available in Selected INT-LAB.Ref , www.ti3.tu-harburg.de/rump/intlab/INTLABref.pdf .Further reference to [13] can be found in *citeseerx.ist.psu.edu* . If we neglect extra work involved in executing Quadratic Backward SOR method the proposed approach studied in the paper is worth the trouble.

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